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Do not open the exam sheet until prompted!

From Molecular to Continuum Physics II — WS 2023/24
Exam — 22.03.2024

Permitted Aids:

- Document-proof writing utensils, but no red pen.
- Two sheets of A4 paper, handwritten on both sides, with name and matriculation number.
- Other aids, especially the use of a calculator, are not allowed.

Hints:

- The use of cell phones during the exam is considered as an attempt to cheat.
- You have a total of **120 minutes** to complete the exam. *All answers must be justified in detail.*
- To pass the exam, **50%** of the possible points are sufficient.
- Please begin each assignment on the sheet on which the assignment is formulated. If you use any of the attached blank sheets in addition to the blank page opposite, please indicate “Continue on another sheet” on the first sheet. *Please mark each sheet with your name and matriculation number — even the blank sheets used.*
- By signing this form, you assure that, to the best of your knowledge, you are fit to take the exam at the beginning of the exam and that the exam performance was performed by you without any unauthorized aids.

Matriculation number: ___ ___ ___ ___ ___ ___

Last name, first name: ______________________________

Signature: ______________________________

Problem	1	2	3	4	5	$\sum$
Points	20	20	20	20	20	100
Your points						

Exam		Bonus		Sum
	+		=	

Grade:

Problem 1.

Consider the incompressible flow of a Newtonian fluid along a pipe of constant radius a with fixed walls. A schematic is shown in Figure 1. The velocity can be written as $\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_z\vec{e}_z$.

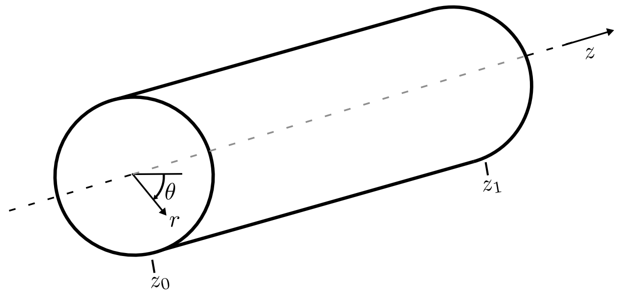

Figure 1: Pipeflow schematic

a) How is the process model called?

b) For a constant pressure gradient $\frac{\partial p}{\partial z} = -P$, is the pressure greater at z_0 or z_1?

c) How do mass and momentum balance read in cylindrical coordinates?

Hint: Gradient and divergence in cylindrical coordinates are given by:

$$\nabla f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{\partial f}{\partial z}\vec{e}_z,$$
$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

d) Assuming constant pressure in the cross-sectional area $\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial r} = 0$, how do mass and momentum balance reduce? What does this imply for the velocity field?

e) Furthermore, there will be no change in the pipe's flow field in time and no flow in radial direction or rotational direction. Hence $u_z = u_z(r,\theta)$ and $u_r = u_\theta = 0$. How does the remaining process model look like?

Hint: In case you were not able to solve the above task, continue task f) and g) with the following equation:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u_z}{\partial r}\right) = -\frac{P}{\mu}r^2$$

f) Derive an explicit expression for the velocity profile. Also, derive an expression that yields the volume discharge?

g) Assume that the pressure grdient drops by a factor of 50%. How can you change the setting to assure the same volume discharge?

20 points

Problem 2.

The shallow water equations with velocity profile correction in one space dimension are given by:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ \alpha h u^2 + g\frac{h^2}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

where $h(x,t)$ describes the fluid depth, $u(x,t)$ the mean flow velocity and $g = 9.81\frac{m}{s^2}$ is the gravitational acceleration.

a) What does α stand for and how can it be computed? Determine α for a linear velocity profile with $u(y=0) = 0$ and $u(y=h) = u_s$.

b) Assume a shallow lake, currently at rest. That means $h = H$ and $u = 0$. A stone is thrown into the lake and causes a small surface perturbation in height and velocity, denoted by h_1 and u_1. Derive a system of equations for h_1 and u_1 by means of linearization.

> **Hint:** The linearization can be performed by looking at small perturbations from a ground state (h_0, u_0), e.g. by defining: $h(x,t) = h_0(x,t) + \delta\, h_1(x,t) + \mathcal{O}(\delta^2)$ and $u(x,t) = u_0(x,t) + \delta\, u_1(x,t) + \mathcal{O}(\delta^2)$, where δ denotes the small amplitude of the perturbation.

c) Reduce the system derived in b) to an equation for h_1. What type of equation do we find?

> **Hint:** For d) and e), in case you were not able to solve the tasks b) and c), please continue with the following equation for h_1:
>
> $$\frac{\partial^2}{\partial t^2} h_1 - gH^2 \frac{\partial^2}{\partial x^2} h_1 = 0$$

d) The wave ansatz is given by $h_1(x,t) = A\, e^{i(k\,x - \omega\, t)}$. State what A, i, k and ω stand for? Using the wave ansatz, compute the phase velocity.

e) Assume that we wait $10s$ until a surface perturbation caused by a stone reaches the lake's shore. How much longer would we have to wait at a similarly sized lake on Saturn's moon Titan? (Titan has surface lakes of liquid ethane and methane and a gravitational acceleration of $g = 1.4\frac{m}{s^2}$)

20 points

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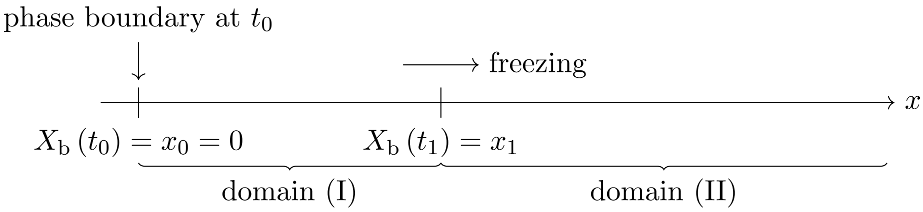
Problem 3.

Let's assume you have a box with cold water at the melting temperature $T_{\mathrm{m}} = 0\,°\mathrm{C}$. You cool down the left side ($x_0 = 0$) of the box at a constant temperature $T_{\mathrm{wall}} = -10\,°\mathrm{C}$ and you are interested in how fast ice grows from the left side with time. If we assume that the density of ice and water is the same, we can model this in 1-D:



a) What is the difference between domain (I) and domain (II)?

b) State the transient heat equation including boundary conditions for both domains. Is the problem already closed? If not, what is the missing condition (both name and equation)? Write down and interpret physically.

c) Show that for fixed interface position X_{b} the following temperature profile

$$T(x,t) = T_{\mathrm{wall}} - \left(T_{\mathrm{wall}} - T_{\mathrm{m}}\right) \frac{\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}{\operatorname{erf}\left(\frac{X_{\mathrm{b}}}{2\sqrt{\alpha t}}\right)}, \tag{2}$$

with the error function defined as

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(-\tilde{z}^2\right) \mathrm{d}\tilde{z},$$

solves the process model in domain (I) as well as the boundary conditions from task b) (not the interface condition).

d) Assume the ansatz

$$X_{\mathrm{b}}(t) = 2\sqrt{\alpha}\lambda\sqrt{t}.$$

Substitute the ansatz and (2) into the interface condition from task b). Simplify the result by using

$$\mathrm{Ste} = \frac{L}{c_p\left(T_{\mathrm{wall}} - T_{\mathrm{m}}\right)},$$

where L is the latent heat of melting, and c_p is the specific heat capacity. Specify the units and interpret Ste physically. Write as an homogeneous function ($F(\lambda) = 0$). Sketch F. How many roots does F have? What is the relevance of the roots?

e) *Qualitatively* sketch the interface position as a function of time for water ice. How would the situation look like for a material that has a thermal diffusivity $\tilde{\alpha}$ larger by a factor of 2, hence $\tilde{\alpha} = 2\alpha$? How would the situation look like for a material that has a heat capacity $\tilde{c_p}$ larger by a factor of 2, hence $\tilde{c_p} = 2c_p$?

20 points

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Problem 4.

Let's assume we have given a heat equation of constant advection speed $-v$:

$$\underbrace{\frac{\partial}{\partial t}T}_{\text{(I)}} - \underbrace{v\frac{\partial}{\partial x}T}_{\text{(II)}} = \underbrace{\lambda\frac{\partial^2}{\partial x^2}T}_{\text{(III)}} \tag{3}$$

on a domain $x = [0, \infty)$ with boundary conditions

$$T|_{x=0} = T_{\text{wall}}, \quad T|_{x\to\infty} = T_{\text{inf}} = \text{const.}, \quad T_{\text{wall}} > T_{\text{inf}}. \tag{4}$$

a) Explain the physical meaning of the terms (I), (II), and (III) of the heat equation.

b) Give the units of t, x, v, and λ.

c) We introduce the dimensionless variables

$$t := t_0\tilde{t},\ x := x_0\tilde{x},\ v := v_0\tilde{v}, \text{and } T := \underbrace{(T_{\text{wall}} - T_{\text{inf}})}_{T_0}\tilde{T} + T_{\text{inf}}.$$

Write both equation (3) and the boundary conditions (4) in dimensionless variables. Identify the Peclet number $\text{Pe} = \frac{v_0 x_0}{\lambda}$.

Now consider a physical regime characterized by $t_0 \gg \frac{x_0^2}{\lambda}$.

d) How does the equation read in this physical regime?

e) Solve for the temperature profile and sketch the solution.

f) How does the profile change as the thermal diffusivity decreases? Indicate it in the sketch!

g) If λ doubles, how does the velocity need to change to retain the same profile?

20 points

Problem 5.

The 1-D elastic wave equation for pressure waves (P-waves) in *velocity-stress formulation* is given by

$$\frac{\partial}{\partial t}\,\underline{\mathbf{Q}} - \underline{\underline{\mathbf{A}}}\,\frac{\partial}{\partial x}\,\underline{\mathbf{Q}} \;=\; \underline{\mathbf{0}}, \tag{5}$$

where

$$\underline{\mathbf{Q}} = \begin{pmatrix} \sigma \\ v \end{pmatrix}, \quad \underline{\underline{\mathbf{A}}} = \begin{pmatrix} 0 & \lambda + 2\mu \\ \rho^{-1} & 0 \end{pmatrix}, \tag{6}$$

with σ denoting the stress and v denoting the particle velocity. λ and μ are the Lamé parameters and ρ is the density. Consider a harmonic plane wave propagating in x-direction given by

$$\underline{\mathbf{Q}}\,(x,t) = \underline{\mathbf{Q}}_0 \exp\left[\mathrm{i}\,(kx - \omega t)\right], \tag{7}$$

with $\underline{\mathbf{Q}}_0 = \begin{pmatrix} \sigma_0 \\ v_0 \end{pmatrix} = \text{const.}$

a) What are the SI base units of σ, v, ω and k?

b) Sketch the real part of $\underline{\mathbf{Q}}\,(x_0, t)$ over time for a fixed position x_0. Name the axes in the sketch. Use the sketch to explain amplitude, period, frequency and phase of the wave.

c) Insert the ansatz (7) into the wave equation (5) and calculate the wave speeds $c = \frac{\omega}{k}$. Explain the physical meaning of your result.

An alternative form of the wave equation is the *displacement formulation* given by

$$\frac{\partial^2 w}{\partial t^2} = \frac{\lambda + 2\mu}{\rho}\frac{\partial^2 w}{\partial x^2}. \tag{8}$$

d) Which of the two formulations is a first order PDE and which formulaiton is a second order PDE? Describe the difference between v and w. If you had to implement a numerical solver using one of the two formulations, which one would you pick? Give reasons for your choice.

20 points

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